Math 434 Assignment 6

Due June 7

Assignments will be collected in class.

1. Prove that for each α, the Σ^0_α and Π^0_α sets are closed under continuous substitution. Conclude that the Borel sets are closed under continuous substitution.

2. Prove that if Λ is a pointclass closed under continuous substitution, then

$$\mathrm{Det}_\omega(\Lambda) \Longrightarrow \mathrm{Det}^*_\omega(\Lambda) \Longrightarrow \mathrm{Det}^*_2(\Lambda).$$

To prove $\mathrm{Det}_\omega(\Lambda) \Longrightarrow \mathrm{Det}^*_\omega(\Lambda)$, use the fact that there is a bijection between ω and $\omega^{<\omega}$. If this bijection is $f : \omega \to \omega^{<\omega}$, define

$$B = \{x \in \omega^\omega : f(x_0)\hat{}\,x_1\hat{}\,f(x_2)\hat{}\,x_3\hat{}\cdots \in A\}.$$

Prove that B is in Λ and that the player that wins $\mathcal{G}_B$ also wins $\mathcal{G}^*_A$.

3. Using the Axiom of Choice, show that:

   (a) there is a set $A \subseteq \omega^\omega$ such that $\mathcal{G}_A$ is determined but $G_{\omega^\omega - A}$ is not determined.

   (b) there are sets $A, B \subseteq \omega^\omega$ such that $\mathcal{G}_A$ and $\mathcal{G}_B$ are determined, but $\mathcal{G}_{A\cup B}$ is not determined.

4. Show that the Axiom of Determinacy implies that there are no ultrafilters on ω.

   Hint: Suppose that there was such an ultrafilter $\mathcal{U}$. Consider the game where players I and II together partition ω into two pieces. On their turn, a player chooses a finite subset of ω, disjoint from all of the subsets previously choosen, to add to their piece. Whichever player's piece ends up in $\mathcal{U}$ is the winner.

   Show that neither player can have a winning strategy. To do this, suppose that I has a winning strategy σ, and find two different ways for II to play against σ so that in one, II builds the piece A, and in the second, II builds the piece B, with $A \cup B$ containing all but finitely many elements of ω.

For the next questions, we will need the notion of a *club*, which is short for closed and unbounded. Let κ be a limit ordinal. A set $C \subseteq \kappa$ is a club on κ if it is unbounded in κ, and closed in the order topology. Being closed in the order topology means that if $\alpha < \kappa$, and $\sup(C \cap \alpha) = \alpha$, then $\alpha \in C$. That is, whenever a sequence in C has a limit that is less than κ, then that limit is in C.

5. Let κ be a regular cardinal. If C and D are clubs, then $C \cap D$ is a club.

   Hint: It is not hard to show that $C \cap D$ is closed. First think about showing that $C \cap D$ is non-empty by considering a sequence $c_0 < d_0 < c_1 < d_1 < c_2 < \cdots$ with $c_i \in C$ and $d_i \in D$.

6. Let $A \subseteq \omega_1$. Define a game $\mathcal{GC}_A$ played on the set $X = \omega_1$ as follows. Players I and II alternate playing ordinals $\xi < \omega_1$ to produce a sequence $\xi_0, \xi_1, \xi_2, \ldots$ with the requirement that $\xi_0 < \xi_1 < \xi_2 < \cdots$. The first player to violate this requirement loses. If both players follow this requirement, then I wins if and only if $\sup \xi_i \in A$.

   (a) Show that I wins $\mathcal{GC}_A$ if and only if A contains a club.

   Hint: The hard direction is to show that if I wins then A contains a club. First show that A is unbounded.

   Any club in ω_1 would have to have order type ω_1. Define by transfinite recursion an increasing sequence of elements $a_\alpha \in A$ for $\alpha < \omega_1$ so that $C = \{a_\alpha : \alpha < \omega_1\}$ is closed (hence a club).

   At limit stages, let $a_\alpha = \sup_{\beta<\alpha} a_\beta$. At successor stages, define $a_\alpha \in A$ to be such that whenever there is a finite partial play of $\mathcal{GC}_A$ where I follows σ and II plays only elements from $\{a_\beta : \beta < \alpha\}$, a_α is larger than any ordinal that appears.

   You must explain why a_α exists for α a successor ordinal, and show that for α a limit ordinal, $a_\alpha \in A$. You must also argue that C is closed. It will naturally be unbounded since it has order type ω_1.

   A similar argument shows that II wins $\mathcal{GC}_A$ if and only if A is disjoint from a club.

   (b) Let $\mathcal{U}$ be the set of all subsets of ω_1 that contain a club. This is called the club filter.

   Assume that for every set $A \subseteq \omega_1$, $\mathcal{GC}_A$ is determined. Show that $\mathcal{U}$ is an ultrafilter.